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LETTER TO THE EDITOR

Two-dimensional percolation: logarithmic corrections to the critical behaviour from series expansions

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Abstract. We have analysed several extant series for the mean cluster size, the zeroth moment of the pair connectedness and the percolation probability for bond and site percolation on two-dimensional lattices with a view to detecting possible logarithmic corrections. The logarithmic correction exponent z is found to be in the range $0^- \leq z \leq 0.15$, and our analysis also provides some new information about the critical exponents γ and β.

Following the suggestion of Andelman and Berker (1981) that the logarithmic factors that have been found for the $q = 4$ state Potts model (Potts 1952) may also be relevant for the $q \to 1$ limit of this model (i.e. the bond percolation problem), Stauffer (1981) conducted a search for them. He re-examined a Monte Carlo calculation (Eschbach *et al* 1981) of the correlation length critical behaviour and found that the correction exponent z (defined in equation (3) below) would be

$$z = 0.06 \pm 0.06$$

not excluding the possibility $z = 0$. Stauffer (1981) further suggested that the discrepancy which may exist between a presumably exact value of γ (the exponent of the mean cluster size $S(p)$) and the γ derived from some series results may be explainable by logarithmic corrections.

This 'exact' value of γ, and a corresponding estimate of β (the exponent of the percolation probability $P(p)$) can be obtained from the conjectures of den Nijs (1979) that y_T (bond percolation) $= 3/4$, and of Nienhuis *et al* (1980) that y_H (bond percolation) $= 91/48$. Using scaling relations (Nightingale and Blote 1980), one obtains

$$\beta = (d - y_H)/y_T = 5/36 = 0.13888\ldots \tag{1}$$

and

$$\gamma = (2y_H - d)/y_T = 43/18 = 2.3888\ldots. \tag{2}$$

The published estimates for γ from series expansions are by no means unanimous. The range of values $\gamma \geq 2.40$, quoted by Stauffer (1981), includes the estimates of Sykes *et al* (1976a) (derived from the low-density expansions of Sykes and Glen (1976)) for the bond series on the honeycomb (HC), square (SQ) and triangular (T) lattices, but not any of their site estimates which are lower. The (low-density, bond series) estimate of Dunn *et al* (1975) for the zeroth moment of the pair connectedness μ_0 (whose critical behaviour is also governed by γ) is

$$\gamma = 2.38 \pm 0.02$$

and the estimate of Domb and Pearce (1976) (from analysis of several bond and site, low- and high-density series) is

$$\gamma \geqslant 2.38 \pm 0.02$$

both compatible with equation (2). In the light of the reservations expressed by Sykes *et al* (1976a) concerning their subjective error estimates, these differences are not surprising, but the hypothesis of logarithmic corrections seems worthy of investigation.

For the exponent β Blease *et al* (1978) obtain

$$\beta = 0.139 \pm 0.003$$

from several high-density series for the T-bond problem, which is consistent with Sykes *et al* (1976b) who found that

$$\beta = 0.138 \pm 0.007$$

from series on several different lattices. These results both agree with equation (1); however the large relative errors in the β values are potentially problematical.

In this work we report on our search for logarithmic corrections in several $S(p)$, $\mu_0(p)$, $P(q)$, $\tilde{P}(q)$ series, where $q \equiv 1-p$, and $P(p)$ is the probability that a given site (if present) belongs to an infinite cluster, whereas $\tilde{P}(p)$ is the probability that a given bond (when present) belongs to an infinite cluster (Blease *et al* 1978). To achieve this objective we assume a behaviour for $S(p)$, $\mu_0(p)$, $P(p)$ and $\tilde{P}(p)$ of the form (for $x < x_c$)

$$f(x) = c(x)(x_c - x)^h \log^{zh}(x_c - x) \tag{3}$$

where h denotes the critical exponent ($-\gamma$ or β) and x denotes p or q. Using a finite number of terms of the power series of $f(x)$ (in powers of x), we derive the power series for

$$g(x) = \frac{1}{h}(x - x_c) \log(x_c - x)\left(\frac{f'(x)}{f(x)} + \frac{h}{x_c - x}\right) \tag{4}$$

and when $c(x)$ is finite at x_c it follows immediately that

$$\lim_{x \to x_c} g(x) = z. \tag{5}$$

We note that equation (5) holds not only in the case where $c(x)$ is analytic at x_c, but also when confluent corrections are present. For example when

$$c(x) \sim c_1[1 + c_2(x_c - x)^{\Delta} + \ldots], \tag{6}$$

where $\Delta > 0$, is the exponent of a possible confluent correction. Such confluent terms were recently shown to exist in the $S(p)$, $\mu_n(p)$ and $P(p)$ series for directed bond percolation (Adler *et al* 1981). Evidence for their presence in isotropic percolation at $d = 2$ and $d = 3$ may be deduced from ε-expansion results and their Padé–Borel resummation (Aharony 1980, Houghton *et al* 1978) and they may be an alternative explanation for any discrepancy that exists between the series estimates for critical exponents and the conjectured 'exact' values. They have also been observed in the series for the generating function for the total number of clusters with s sites (Gaunt *et al*

1976) and in the series for $P_c(\lambda)$, the percolation analogue of the magnetic field variation of the magnetisation along the critical isotherm (Gaunt and Sykes 1976).

In the presence of confluent corrections equations (4) and (6) give

$$g = z + \frac{c_2\Delta}{h}\frac{(x_c - x)^{\Delta}\log(x_c - x)}{1 + c_2(x_c - x)^{\Delta}} + \dots$$

but since $\Delta > 0$ equation (5) still holds.

Evaluation of $g(x \to x_c)$ is carried out by forming Padé approximants to $g(x)$. The required input is the values of x_c and h, and in our preliminary studies we found that the value of z is rather sensitive to the input x_c. Therefore, we restrict ourselves to the problems where p_c is known exactly, namely the SQ lattice bond ($p_c = \frac{1}{2}$), the T lattice site ($p_c = \frac{1}{2}$) and bond ($p_c = 1 - 2\sin(\pi/18)$), and the HC lattice bond ($p_c = 2\sin(\pi/18)$) percolation problems. When the leading exponent h is varied, each Padé approximant $[N, K]$ (where N and K are the powers of the numerator and the denominator respectively) defines a curve $z(h)$.

For all four cases mentioned above we have studied the low-density series for the mean cluster size ($S(p)$, $x = p$, $h = -\gamma$) taken from Sykes and Glen (1976). For the T lattice bond problem, we have also investigated the series for the zeroth moment of the pair connectedness ($\mu_0(p)$, $x = p$, $h = -\gamma$), of Dunn *et al* (1975).

In figures 1, 2 and 3 we summarise the $z(\gamma)$ curves that were obtained from the central (largest $N + K$ and closest to diagonal) Padé approximants for the SQ-bond $S(p)$, T-site $S(p)$ and T-bond $\mu_0(p)$ series respectively. We find that in each case there is a central 'grouping' of the majority of Padé approximants. This central group of $z(\gamma)$ functions possesses surprising similarity when we compare the $S(p)$ bond and site results, and the $S(p)$ and $\mu_0(p)$ results—an astonishing manifestation of universality.

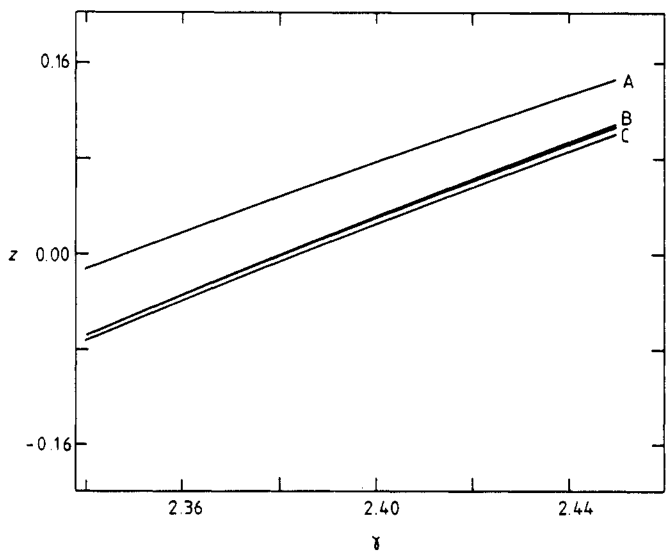

Figure 1. Plot of the Padé approximants to the logarithmic correction factor z as a function of γ for the $S(p)$ series for the bond percolation problem on the square lattice. Curve A indicates the [5, 8] approximant, B indicates the central group of [6, 7], [7, 6], [5, 7], [6, 6], [7, 5], [5, 6] and [6, 5] approximants and C indicates the [8, 5] approximant.

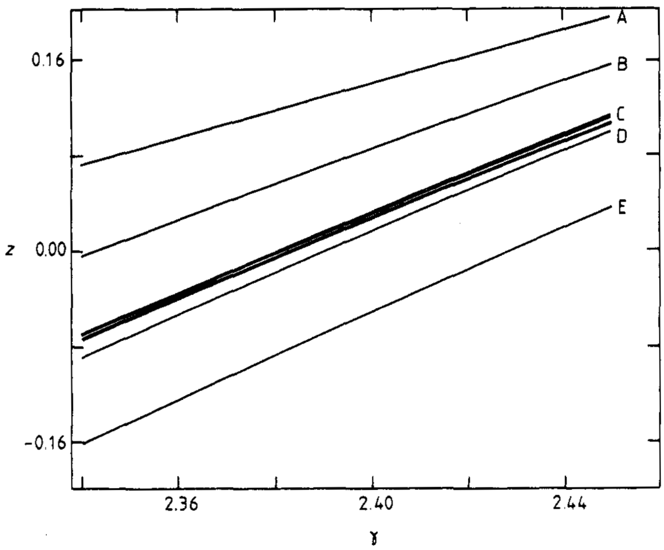

Figure 2. Plot of the Padé approximants to the logarithmic correction factor z as a function of γ for the $S(p)$ series for the site percolation problem on the triangular lattice. The curves A and B indicate the [6, 8] and [5, 9] approximants respectively, C indicates the central group of [7, 7], [8, 6], [9, 5], [7, 6] and [8, 5] approximants, and D and E indicate the [5, 8] and [6, 7] approximants respectively.

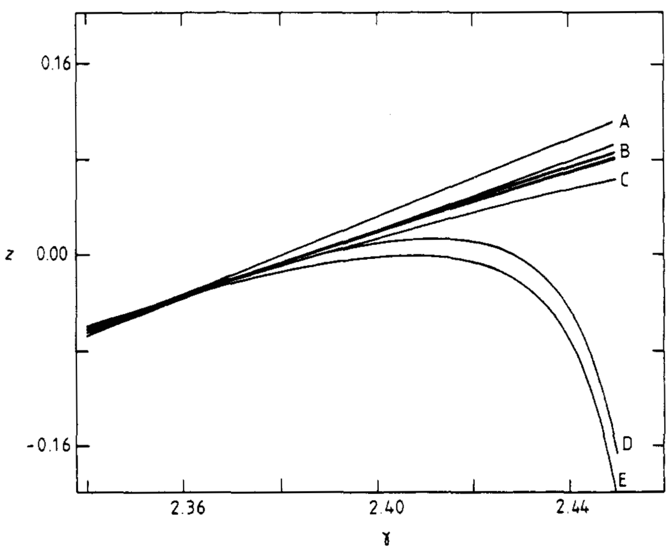

Figure 3. Plot of the Padé approximants to the logarithmic correction factor z as a function of γ for the $\mu_0(p)$ series for the bond percolation problem on the triangular lattice. Curve A indicates the [4, 5] approximant (which exhibited a weak pole at $\gamma \sim 2.352$ which is not reproduced in the figure), and curves B indicate a central group of [5, 4], [6, 3], [4, 4], [5, 3] and [4, 3] approximants. Curve C indicates the [3, 5] approximant, and D and E indicate the [3, 6] and [3, 4] approximants respectively.

Judging from the spread within this central group we are able to determine a range of z values to correspond to the different choices of γ discussed above. We have

$$\gamma = 2.38 \pm 0.02 \rightarrow -0.04 \leqslant z \leqslant 0.03$$

$$\gamma = 2.40 \pm 0.03 \rightarrow -0.02 \leqslant z \leqslant 0.08$$

and

$$\gamma = 2.43 \pm 0.03 \rightarrow 0.02 \leqslant z \leqslant 0.15$$

and note for purposes of comparison that for these three series the 'direct' estimates (using Dlog Padé and extrapolation methods) were found to be $\gamma = 2.425 \pm 0.005$, $\gamma = 2.40 \pm 0.03$ and $\gamma = 2.38 \pm 0.02$ (Sykes *et al* 1976a, Dunn *et al* 1975) respectively. If we accept the conjectured 'exact' value of γ (equation (2), $\gamma = 43/18$) we find that the z values vary in the ranges $0.012 \leqslant z \leqslant 0.013$, $0.011 \leqslant z \leqslant 0.014$ and $0.000 \leqslant z \leqslant 0.005$ respectively for these three series. Finally, we examined the γ values for which $z(\gamma) = 0$ for the different central curves. For all three series these estimates fell in the range

$$2.380 \leqslant \gamma \leqslant 2.389.$$

It must be stressed that the ranges of γ and z values which we present here should not be considered as absolute error bounds, but rather as the ranges of spread of the different Padé approximant values. Not only do we not possess a more reliable method of error estimates for this type of analysis, but we are unable to exclude the danger of systematic errors. Particularly, since we are evaluating Padé approximants to the function $g(x)$ (equation (4)) which has *logarithmic terms*, we cannot expect the procedure to be convergent in general. For example, in the case of the T-bond $S(p)$ series (for which the $z(\gamma)$ curves are summarised in figure 4) we found a spread over a larger

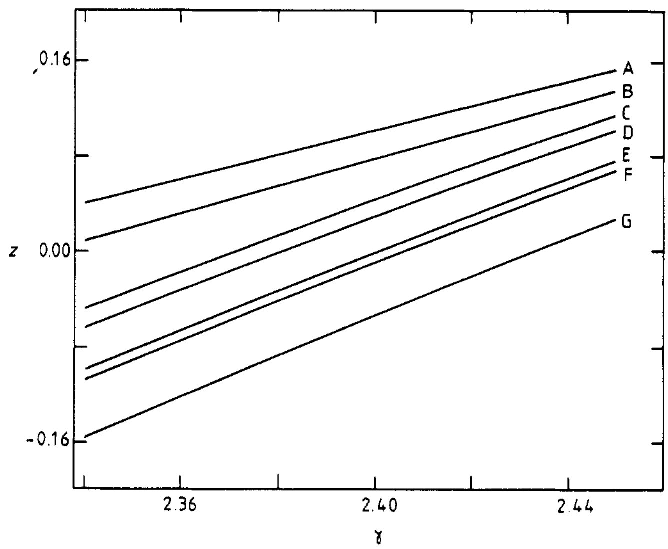

Figure 4. Plot of the Padé approximants to the logarithmic correction factor z as a function of γ for the $S(p)$ series for the bond percolation problem on the triangular lattice. Curves A–G indicate the [3, 5], [3, 6], [4, 5], [6, 3], [4, 4], [5, 3] and [4, 3] approximants respectively. (The [5, 4] and [3, 4] approximants extended above the scale and are not plotted.)

range of z values, and no central 'grouping'. Because of this wider spread, the only useful information we may obtain from figure 4 is that the z values are consistent with the preceding analysis.

Our results for the HC-bond $S(p)$ series resemble the T-lattice bond case (wide spread of $z(\gamma)$ curves and no 'central' grouping); again there is no inconsistency with the preceding estimates, and thus we do not present a graph.

Having noted that the convergent estimates are extremely consistent and the less convergent $z(\gamma)$ curves in no way contradict their behaviour, we may intimate from our above analysis that z is zero or close thereto for input γ values near the conjectured 'exact' value. In turn, the γ values obtained with the assumption $z \equiv 0$ are consistent with most of the 'direct' estimates. There remains, however, the range of values, $\gamma \sim 2.42$, quoted by Sykes *et al* (1976a) which are not consistent with the above picture, and imply $z > 0$. For both regions we may estimate that $0 \leqslant z \leqslant 0.15$, a range of values similar to the range $0 \leqslant z \leqslant 0.12$ obtained by Stauffer (1981) for the correlation length.

We now turn to the percolation probabilities $P(p)$ and $\tilde{P}(p)$, which are expected to have the same critical behaviour, with an exponent β. The available series in this case are high-density ($x = q$) series for $P(q)$ for all four lattice and site/bond permutations considered above and $\tilde{P}(q)$ for the T-bond problem (Sykes *et al* 1976b, Blease *et al* 1978).

The results for the T-bond $\tilde{P}(q)$ series are plotted in figure 5. We find a clear central 'grouping' of $z(\beta)$ curves, and for the relevant β values we obtain

$$\beta = 0.139 \pm 0.007 \rightarrow -0.2 \leqslant z \leqslant 0.3$$

$$\beta = 0.138 \pm 0.003 \rightarrow -0.05 \leqslant z \leqslant 0.15$$

and

$$\beta = \tfrac{5}{36} = 0.1388\ldots \rightarrow 0.09 \leqslant z \leqslant 0.13.$$

The range of β values for which the 'central' $z(\beta)$ curves vanish is

$$z = 0 \rightarrow 0.1345 \leqslant \beta \leqslant 0.1365.$$

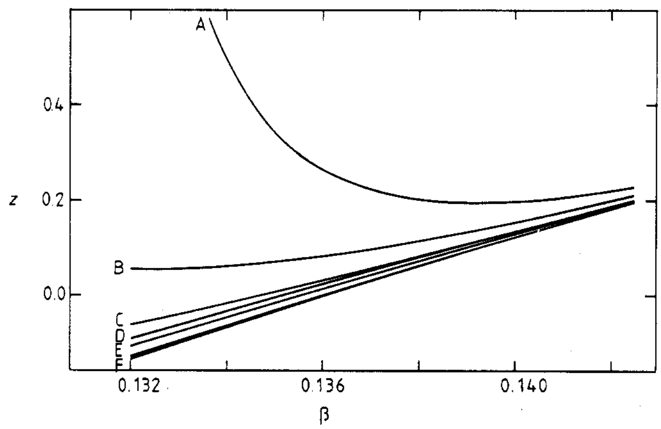

Figure 5. Plot of Padé approximant estimates of the correction factor z as a function of β for the $\tilde{P}(q)$ series for the bond percolation problem on the triangular lattice. Curves A–E indicate the [15, 14], [14, 15], [17, 13], [15, 15] and [16, 13] approximants respectively, and curves F indicate the group of [13, 17], [14, 16], [16, 14] and [13, 16] approximants.

If we consider the narrower range of β, namely $\beta = 0.138 \pm 0.003$ (Blease *et al* 1978), the corresponding range of z values is observed to be similar to the $S(p)$ and $\mu_0(p)$ cases, and to the values of Stauffer (1981). However, in the $\tilde{P}(q)$ case the conjectured 'exact' value of $\beta = 5/36$ is inconsistent with $z = 0$.

The results for the $P(q)$ series for T, SQ, HC-bond and T-site problems are less conclusive. We found a considerable spread of $z(\beta)$ curves in these cases. A typical situation is that of the T-bond $P(q)$ series, plotted in figure 6. Here the $z(\beta)$ values are consistent with the results for the $\tilde{P}(q)$ series, although the uncertainties are much larger in the $P(q)$ case, and it must be stressed, that again $\beta = 5/36$ and $z = 0$ are inconsistent. For all four $P(q)$ series we find that

$$\beta = 5/36 \rightarrow 0.1 \leqslant z \leqslant 0.5$$

and thus $z(5/36)$ is clearly positive.

In summary, we have found that both the high-density and the low-density series suggest that

$$0^- \leqslant z \leqslant 0.15$$

which is consistent with the Monte Carlo results of Stauffer (1981) and we found that the assumption that $z = 0$ would exclude some γ estimates as well as the conjectured β value.

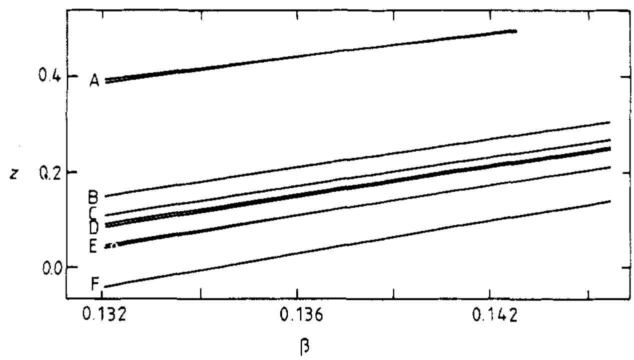

Figure 6. Plot of Padé approximant estimates of the correction factor z as a function of β for the $P(q)$ series for the bond percolation problem on the triangular lattice. Curves A indicate the [13, 17] and [13, 16] approximants, and curves B and C indicate the [14, 15] and [16, 13] approximants respectively. Curves D and E indicate the [15, 15] and [16, 14] and the [17, 13] and [15, 14] approximants respectively, and curve F indicates the [14, 16] approximant.

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